

Production Functions

The production Function shows the relationship between input and output.

$$Q = f(L, K, L_a, E)$$

where:

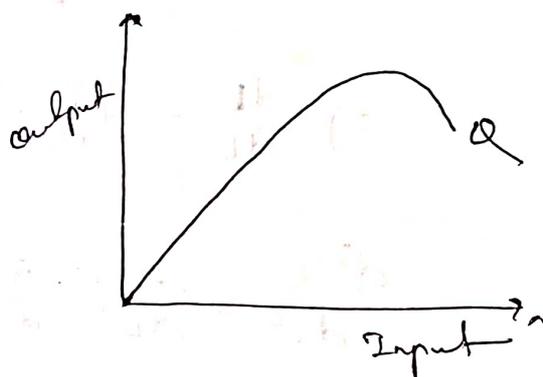
Q = Output

L = Labour

K = Capital

L_a = Land

E = Entrepreneurship



Cobb-Douglas Prodⁿ Functions

- It is developed by Charles Cobb & Paul Douglas in 1930's.

$$Y = AK^\alpha L^{1-\alpha} \quad (\beta)$$

where Y: Output

L = Labour

K = Capital

A = ~~total~~ level of Technology

α = constant ($0 < \alpha < 1$)

Features

1- It is homogeneous of degree $(\alpha + \beta)$. In the special case of $\alpha + \beta = 1$, the function turns into linearly homogeneous function and will depict constant returns to scale.

$$Q = A(tK)^\alpha (tL)^\beta = A t^\alpha K^\alpha t^\beta L^\beta$$

$$= A t^{\alpha+\beta} K^\alpha L^\beta = t^{\alpha+\beta} (AK^\alpha L^\beta)$$

$$= t^{\alpha+\beta} (Q)$$

2- Isoquants are negatively sloped and convex downwards.

$$Q = f(L, K) = AK^\alpha L^\beta$$

$$dQ = \frac{\partial Q}{\partial K} \cdot dK + \frac{\partial Q}{\partial L} \cdot dL$$

$$\frac{\partial Q}{\partial K} = A\alpha K^{\alpha-1} L^\beta$$

$$\frac{\partial Q}{\partial L} = A\beta K^\alpha L^{\beta-1}$$

Since output remains unchanged on a particular isoquant, the total change in output $dQ = 0$.

$$dQ = (A\alpha K^{\alpha-1} L^{\beta}) dk + (A\beta K^{\alpha} L^{\beta-1}) dL = 0.$$

$$\Rightarrow \frac{dk}{dL} = -\left(\frac{\beta}{\alpha} \cdot \frac{k}{L}\right) \text{ (isoquants have - slope).}$$

$$\frac{d^2k}{dL^2} = \frac{d}{dL} \left(-\frac{\beta}{\alpha} \cdot \frac{k}{L}\right) = -\frac{\beta}{\alpha} \cdot \frac{d}{dL} \left(\frac{k}{L}\right)$$

$$= -\frac{\beta}{\alpha} \cdot \frac{1}{L^2} \left(L \cdot \frac{dk}{dL} - k\right) \text{ (Quotient Rule).}$$

$$= -\frac{\beta}{\alpha} \cdot \frac{1}{L} \cdot \frac{dk}{dL} + \frac{\beta}{\alpha} \cdot \frac{k}{L^2}$$

$\frac{dk}{dL}$ has negative sign, therefore, whole expansion of

$\frac{d^2k}{dL^2}$ will be positive (as α, β, L and k can never take negative values).

$$\frac{dk}{dL} < 0 \text{ and } \frac{d^2k}{dL^2} > 0. \text{ (convex downwards).}$$

(3) Its exponents α and β represent:

- Elasticity of output with respect to total capital and labour.

$$\text{Labour Elasticity} = \frac{L}{Q} \cdot \frac{\partial Q}{\partial L}$$

$$\frac{L}{Q} \cdot \frac{\partial Q}{\partial L} = \frac{L}{AK^{\alpha}L^{\beta}} \cdot A \cdot \beta \cdot K^{\alpha} L^{\beta-1} = \beta$$

$$\text{Capital Elasticity} = \frac{k}{Q} \cdot \frac{\partial Q}{\partial K}$$

$$\frac{k}{AK^{\alpha}L^{\beta}} \cdot A\alpha K^{\alpha-1} L^{\beta} = \alpha.$$

— Share of each input in total output.

$$\frac{\partial Q}{\partial K} = A \alpha K^{\alpha-1} L^{\beta}$$

$$\frac{\partial Q}{\partial K} = \frac{\alpha}{K} A K^{\alpha} L^{\beta}$$

$$= \alpha \cdot \frac{Q}{K}$$

$$= \alpha (APK)$$

$$\frac{\partial Q}{\partial K} = MPK$$

AP: Average product

MP: Marginal product

$$MPK = \alpha \cdot (APK)$$

Similarly,

$$MP_L = \beta \cdot (AP_L)$$

Marginal product of labour should equate with real wage rate. $MP_L = \frac{w}{P}$

$$MP_L = \beta \cdot \frac{Q}{L} = \frac{w}{P}$$

$$\beta = \frac{wL}{QP}$$

→ Percentage share of labour in total output.

$$\text{Similarly } \Rightarrow \alpha = \frac{rK}{QP}$$

(r → Price for per unit of Capital)

→ Percentage share of Capital in total output.